EPJ manuscript No.
(will be inserted by the editor)

arXiv:math/0609626v2 [math.DS] 29 Jan 2007

Evolutionary Prisoner's Dilemma on heterogeneous Newman-Watts small-world network

F. Fu[a], L.-H. Liu, and L. Wang[b]

Center for Systems and Control, College of Engineering, Peking University, Beijing 100871, P. R. China

Received: date / Revised version: date

Abstract. In this paper, we focus on the heterogeneity of social networks and its role to the emergence of prevailing cooperation and sustaining cooperators. The social networks are representative of the interaction relationships between players and their encounters in each round of games. We study an evolutionary Prisoner's Dilemma game on a variant of Newman-Watts small-world network, whose heterogeneity can be tuned by a parameter. It is found that optimal cooperation level exists at some intermediate topological heterogeneity for different temptations to defect. That is, frequency of cooperators peaks at some specific values of degree heterogeneity — neither the most heterogeneous case nor the most homogeneous one would favor the cooperators. Besides, the average degree of networks and the adopted update rule also affect the cooperation level.

PACS. 02.50.Le Decision theory and game theory – 89.75.Hc Networks and genealogical trees – 87.23.Ge Dynamics of social systems

1 Introduction

Evolutionary game theory has been well developed as an interdisciplinary science by researchers from biology, economics, social science, computer science for several decades. In past few years, it also gained the interests of physicists to study some phenomena and intriguing mechanisms in well-mixed population by using mean-field theory of statistical physics. In classic game theory, the players are assumed to be completely rational and try to maximize their utilities according to opponents' strategies. Under these assumptions, in the Prisoner's Dilemma game (PD game), two players simultaneously decide whether to cooperate (C) or to defect (D). They both receive R upon mutual cooperation and P upon mutual defection. A de-

[a] e-mail: fufeng@pku.edu.cn

[b] e-mail: longwang@pku.edu.cn

fector exploiting a C player gets T, and the exploited cooperator receives S, such that $T > R > P > S$ and $2R > T + S$. As a result, it is best to defect regardless of the co-player's decision. Thus, defection is the evolutionarily stable strategy (ESS), even though all individuals would be better off if they cooperated. Thereby this creates the social dilemma, because when everybody defects, the mean population payoff is lower than that when everybody cooperates. However, cooperation is ubiquitous in natural systems from cellular organisms to mammals. In the past two decades, some extensions on PD game have been considered to elucidate the underlying mechanisms boosting cooperation behaviors by which this dilemma could be resolved. For instance, depart from the well-mixed population scenario, Nowak and May considered PD game on spatially structured population [1]. All individuals are constrained to play only with their immediate neighbors. They found that spatial structure enhances the ability of cooperators to resist invasion by defectors.

The successful development of network science provides a systematic framework for studying the dynamics processes taking place on complex networks [2,3]. For most networks, including the World Wide Web, the Internet, and the metabolic networks, they have small-world property and are demonstrated to be scale-free [4,5]. The network theory is a natural and convenient tool to describe the population structure on which the evolution of cooperation is studied. Each vertex represents an individual and the edges denote links between players in terms of game dynamical interaction. Trivially, one could conclude that both well-mixed population and spatially-structured population are represented by regular graphs. It is, however, unrealistic to assume the real world is as homogeneous as regular graphs. Furthermore, as aforementioned, many real-world networks of interactions are heterogeneous, namely, different individuals have different numbers of average neighbors with whom they interact. In [6], Abramson and Kuperman studied a simple model of an evolutionary version of the Prisoner's Dilemma game played in small-world networks. They found that defectors prevail at some intermediate rewiring probability. By introducing the volunteering participation, Szabó *et al.* investigated the spatio-temporal diagrams of the evolutionary process within the context of rewired lattices which have small-world property [7]. Other paradigms of game with spatially-structured population or on graphs, such as snowdrift game, rock-scissors-paper game, public goods game, etc, have been studied [8,9,10].

Recently, Santos and Pacheco discovered that in scale-free networks, the cooperators become dominant for entire range of parameters in evolutionary PD game and snowdrift game [11]. They also demonstrated that the enhancement of cooperation would be inhibited whenever the correlations between individuals are decreased or removed. Being similar to Hamilton's rule, Ohtsuki *et al* described a simple rule for evolution of cooperation on graphs and social networks — the benefit-to-cost ratio exceeds the average number of neighbors [12]. In [13,14], Santos suggested that the heterogeneity of the population structure, especially scale-free networks, provides a new mechanism for

cooperators to survive. Moreover, cooperators are capable of exploiting the heterogeneity of the network of interactions, namely, finally occupy the hubs of the networks. Besides, Vukov and Szabo studied the evolutionary PD game on hierarchical lattices and revealed that the highest stationary frequency of cooperators occurs in some intermediate layers [15]. Some researchers have studied minority game [16] , with small-world interactions [17], with emergent scale-free leadership structure [18], etc. Howbeit, these works are mostly studied on crystalized (static) networks, i.e., the topology of network is not affected by the dynamics on it. Therefore, adapting the network topology dynamically in response to the dynamic state of the nodes should be taken into account. Evolutionary games with "adaptive" network structure have been investigated by some researchers [19,20].

In this paper, we consider evolutionary PD game on a variant of Newman-Watts (NW) small-world network. As is known, the NW small-world network is moderately homogeneous. In order to investigate the heterogeneity's role in the emergence of cooperation for different temptations to defect, some artificial heterogeneity in network degree sequences is introduced. Namely, some nodes are randomly chosen as hubs and at least one endpoint of each of the added m shortcuts is linked to these hubs. The remainder of the paper is organized as follows. Section II discusses the heterogeneous Newman-Watts (HNW) model and the evolutionary PD game, including payoff matrix and the microscopic updating rule. And then Sec. III gives out the simulation results for different parameters and makes some explanations to the results. Conclusions are made in Sec. IV.

2 The model

2.1 The Heterogeneous Newman-Watts (HNW) model

We consider one-dimensional lattice of N vertices with periodic boundary conditions, i.e., on a ring, and join each vertex to its neighbors κ or fewer lattice spacing away. Instead of rewiring a fraction of the edges in the regular lattice as proposed by Watts and Strogatz [4], some "shortcuts" connecting randomly chosen vertex pairs are added to the low-dimensional lattice [21]. To introduce certain type of heterogeneity to this NW model [22], we choose N_h vertices at random from all N nodes with equal probabilities and treat them as hubs. Then we add each of m shortcuts by connecting one node at random from all N nodes to another node randomly chosen from the N_h centered nodes (duplicate connections and self-links are voided), see Fig. 1 as an illustration. In such heterogeneous Newman-Watts (HNW) model, nodes can be naturally divided into two groups: hubs, which tend to have higher connectivity, and the others that have lower connectivity. Accordingly, the heterogeneity is controlled by the parameter N_h : small N_h leads to higher degree of the hubs, which in turn results in increased heterogeneity. Trivially, one could see that when $N_h = N$, shortcuts are just simply added uniformly at random, effectively reducing the network to the homogeneous Newman-Watts

model. If $N_h = 1$, all shortcuts are linked to a single center, making the network an extremely heterogeneous one. The degree of heterogeneity of the network is computed as $h = N^{-1}\sum_k k^2 N(k) - \langle k\rangle$ (the variance of the network degree sequcence), where $N(k)$ gives the number of vertices with k edges, $\langle k\rangle$ denotes the average degree. From Fig. 2, it shows that variance $\sigma^2 (= h)$ monotonically decreases as the fraction number of hubs N_h/N increases. Thus we simply use the fraction N_h/N to indicate the degree of heterogeneity.

2.2 Evolutionary Prisoner's Dilemma (PD) game

Since the pioneering work on iterated games by Axelrod [23], the evolutionary Prisoner's Dilemma (PD) game has been a general metaphor for studying the cooperative behavior. In the evolutionary PD game, the individuals are pure strategists, following two simple strategies: cooperate (C) and defect (D). The spatial distribution of strategies is described by a two-dimensional unit vector for each player x, namely,

$$s = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{1}$$

for cooperators and defectors, respectively. Each individual plays the PD game with its "neighbors" defined by their who-meets-whom relationships and the incomes are accumulated. The total income of the player at the site x can be expressed as

$$P_x = \sum_{y\in\Omega_x} s_x^T M s_y \tag{2}$$

where the Ω_x denotes the neighboring sites of x, and the sum runs over neighbor set Ω_x of the site x. Without loss of the generic feature of PD game, the payoff matrix has a rescaled form, suggested by Nowak and May [24]

$$M = \begin{pmatrix} 1 & 0 \\ b & 0 \end{pmatrix} \tag{3}$$

where $1 < b < 2$. In evolutionary games the players are allowed to adopt the strategy of one of their more successful neighbors. Then the individual x randomly selects a neighbor y for possibly updating its strategy. Since the success is measured by the total payoff, whenever $P_y > P_x$ the site x will adopt y's strategy with probability given by [11]

$$W_{s_x \leftarrow s_y} = \frac{P_y - P_x}{b k_>} \tag{4}$$

where $k_>$ is the largest between site x's degree k_x and y's k_y. This microscopic updating rule is some kind of imitation process that is similar to Win-Stay-Lose-Shift rules in spirit. Unlike the case of bounded rationality, such proportional imitation (the take-over probability is proportional to the payoff difference of the two sites) does not allow for an inferior strategy to replace a more successful one.

3 Simulation results and discussions

Evolutionary PD game is performed analogically to replicator dynamics: in each generation, all directly connected pairs of individuals x and y engage in a single round game and their accumulated payoffs are denoted by P_x and P_y respectively. The synchronous updating rule is adapted here. Whenever a site x is updated, a neighbor y is chosen at random from all k_x neighbors. The chosen neighbor

takes over site x with probability $W_{s_x \leftarrow s_y}$ as Eq. (4), provided that $P_y > P_x$. Simulations were carried out for a population of $N = 2001$ players occupying the vertices of the HNW network. Initially, an equal percentage of cooperators and defectors was randomly distributed among the population. We confirm that different initial conditions do not qualitatively influence the equilibrium results. In Fig. 3, with fixed chosen hub numbers $N_h = 41$, the equilibrium frequencies of cooperators with respect to different b are almost the same when started from different initial conditions. Equilibrium frequencies of cooperators were obtained by average over 2000 generations after a transient time of 10000 generations. The evolution of the frequency of cooperators as a function of b and N_h/N has been computed. Furthermore, each data point results from averaging over 100 simulations, corresponding to 10 runs for each of 10 different realizations of a given type of network of contacts with the specified parameters. In the following simulations, $N = 2001, \kappa = 2$ are kept invariant.

To investigate the influence of network degree heterogeneity to the evolution of cooperation, we fixed the value of b when increased fraction of hubs N_h/N from $1/N$ to 1. Fig. 4 shows the equilibrium frequency of cooperators as function of the fraction number of hubs N_h/N by different values of b for comparison when $m = 1000$. We find the ρ_c's curve exhibits non-monotonous behavior with a peak at some specific value of N_h/N. As aforementioned, the quantity N_h/N can be regarded as a measurement of heterogeneity. For certain b, it has been illustrated in Fig. 4, there is a clear maximum frequency of cooperators around small N_h/N near the zero. In Fig. 5, we report the result of frequency of cooperation as function of the parameters space $(b, N_h/N)$. It also shows that for fixed b, optimum cooperation level exits at certain intermediate N_h/N. In addition, for certain intermediate N_h/N, the density of cooperation is higher than the cases when $N_h/N \to 0$ or $N_h/N \to 1$. As a result, it reveals that optimal cooperation level occurs at intermediate topological heterogeneity for some fixed b, that's neither the most heterogeneous case nor the most homogeneous one favors the cooperators best. Actually, under the most homogeneous case of $N_h/N = 1$, the cooperation behavior is almost extinct for large values of b. While under the most heterogeneous case of $N_h/N = 1/N$, the cooperation level is also low. The reason for this is that, in the most heterogeneous case, all shortcuts are linked to a single node, making the network star-like with sparse connections between the neighbor sites of the centered node. The peripheral nodes' imitation strategies heavily depend upon the centered individual's choice. Once the unique largest hub is occupied by a defector, defection will spread over the entire network. Then the system can hardly recover from the worst case of all defectors. Therefore the most heterogeneous case with $N_h/N = 1$ is not the ideal case for arising cooperation. The most homogeneous case with $N_h/N = 1$ does not favor cooperation either. In this situation, all nodes almost have the same number of neighbors with some long range shortcuts making the average length short. Therefore, the defection of some node is easily spread from one node to another through the shortcuts.

Within some generations, the defectors prevail on the network. Hence the most homogeneous case with $N_h/N = 1$ does not promote the evolution of cooperation.

Let's consider the case of intermediate heterogeneity where some certain number of nodes are selected as hubs (density of cooperators peaks at some specific value N_h/N for fixed b). These hubs are connected to each other by shortcuts or placed on the same ring linked by the regular edges. This provides the protection to each other in resisting the invasion of defectors. Even if one hub is occupied by the defector, the cooperation level will temporarily decrease due to the diffusion of defection from the hub. However, other hubs occupied by cooperators could reciprocally help each other and promote cooperation behaviors on the network. Finally, cooperation is recovered to the normal level. Accordingly, it in part result from the ability of cooperators taking advantage of hubs. It is reported in [11] that cooperation dominates in Barabási-Albert (BA) scale-free network due to the degree heterogeneity where cooperators are more capable to occupy the hubs in the network. Though the cooperation level on our HNW model is not as high as it on the BA scale-free network (see Fig. 6 and the figure in ref. [11] for comparison), both of them benefit from the heterogeneity of the network in the evolution of cooperation.

The average degree of HNW network $\langle k\rangle = 2\kappa+2m/N$. It is believed that average degree affects the evolution of cooperation. Fig. 7 plots the frequency of cooperators versus N_h/N corresponding to different shortcuts number $m = 800, 1000, 1200$ respectively with $b = 1.1$. We find that increasing average degree (larger m) promotes cooperation on the network. Nevertheless, we should point out that with sufficient large m which makes the network nearly fully-connected, the cooperation will be inhibited due to mean-field behavior [25]. On the other hand, when m changes, our above conclusion on the effect of heterogeneity to cooperation is still valid. We can see in Fig. 7 that, for different fixed m and b, ρ_C still peaks at certain intermediate value of N_h/N.

We further explore the situation where we adopt another microscopic update rule different from Eq. 4. In stead of updating the strategies according to the accumulated payoff of individuals, we consider using the average payoff of individual x, $\overline{P}_x = \frac{P_x}{k_x}$. Thus the new update rule is:

$$W_{s_x \leftarrow s_y} = \frac{\overline{P}_y - \overline{P}_x}{b} \tag{5}$$

By such update rule, we report the results in Fig. 8 corresponding to $b = 1.1$ and $b = 1.3$. It is found that the equilibrium frequency of cooperation is not sensitive to the heterogeneity as N_h/N increases from $1/N$ to 1 (the curves of average payoff are almost flat). It is because that normalizing individual's payoff diminish the role of heterogeneity. Consequently, the difference in individual's accumulated payoff arising from the degree heterogeneity will be positive to evolution of cooperation. We should be careful to come to a conclusion with different update rules. Moreover, all these phenomena and conclusions are also valid under the cases for different population size N.

4 Conclusion remarks and future work

In conclusion, we have studied the effect of topological heterogeneity to the evolution of cooperation behavior in evolutionary Prisoner's Dilemma game. It is found that frequency of cooperation peaks at some specific value of N_h/N, that is, neither the most heterogeneous case nor the most homogeneous one would favor cooperators. We found that the network degree heterogeneity is one of the factors affecting the emergence of cooperation. Besides, the average degree and the adopted update rule also play a role to cooperation. Therefore, it is meaningful and necessary to explore the underlying factors that do matter the emergence of cooperation behaviors. In addition, the interplay between game dynamics and network topologies is an interesting topic for further investigations.

Acknowledgments

F. F. would like to acknowledge Jie Ren for stimulating discussions. The authors are partly supported by National Natural Science Foundation of China under Grant Nos.10372002 and 60528007, National 973 Program under Grant No.2002CB312200, National 863 Program under Grant No.2006AA04Z258 and 11-5 project under Grant No.A2120061303.

References

1. M. A. Nowak and R. M May, Nature **359**, (1992) 826.
2. R. Albert and A.-L. Barabási, Rev. Mod. Phys. **74**, (2002) 47.
3. M. E. J. Newman, SIAM Review **45**, (2003) 167.
4. D. J. Watts and S. H. Strogatz, Nature **393**, (1998) 440.
5. A.-L. Barabási and R. Albert, Science **281**, (1999) 509.
6. G. Abramson and M. Kuperman, Phys. Rev. E **63**, (2001) 030901(R).
7. G. Szabó and J. Vukov, Phys. Rev. E **69**, (2004) 036107.
8. M. Tomassini, L. Luthi, and M. Giacobini, Phys. Rev. E **73**, (2006) 016132.
9. A. Szolnoki and G. Szabó, Phys. Rev. E **70**, (2004) 037102.
10. G. Szabó and C. Hauert, Phys. Rev. Lett. **89**, (2002) 118101.
11. F. C. Santos and J. M. Pacheco, Phys. Rev. Lett. **95**, (2005) 098104.
12. H. Ohtsuki, C. Hauert, E. Lieberman and M. A. Nowak, Nature **441**, (2006) 502.
13. F. C. Santos and J. M. Pacheco, J. Evol. Biol. **19**, (2006) 726.
14. F. C. Santos, J. M. Pacheco and Tom Lenaerts, Proc. Natl. Acad. Sci. USA **103**, (2006) 3490.
15. J. Vukov and G. Szabó, Phys. Rev. E **71**, (2005) 036133.
16. D. Challet and Y.-C. Zhang, Physica A **246**, (1997) 407.
17. M. Kirley, Physica A **365**, (2006) 521.
18. M. Anghel, Z. Toroczkai, K. E. Bassler and G. Korniss, Phys. Rev. Lett. **92**, (2004) 058701.
19. M. G. Zimmermann and V. M. Eguíluz, Phys. Rev. E **72**, (2005) 056118.
20. J. Ren, X. Wu, W.-X. Wang, G. Chen, and B.-H. Wang, arXiv:physics/0605250.
21. M. E. J. Newman and D. J. Watts, Phys. Lett. A **263**, (1999) 341.
22. T. Nishikawa, A. E. Motter, Y.-C. Lai and F. C. Hoppensteadt, Phys. Rev. Lett. **91**, (2003) 014101.

23. R. Axelrod and W. D. Hamilton, Science **211**, (1981) 1390.
24. M. A. Nowak, S. Bonhoeffer and R. M. May, Int. J. Bifurcation Chaos, **3**, (1993) 35.
25. C.-L. Tang, W.-X. Wang, X. Wu, and B.-H. Wang, Eur. Phys. J. B, **53**, (2006) 411.

Fig.1 Illustration of the heterogeneous Newman-Watts small-world model. A one-dimensional lattice with connections between all vertex pairs separated by κ or fewer lattice spacing, with $\kappa = 2$ in this case. The red dots are chosen as centers or hubs and the added m shortcuts (the dashed line) have at least one endpoint belonging to these hubs.

Fig.2 The log-log plot of the variance of network degree sequence versus fraction of hubs N_h/N for $N = 2001, m = 1000$. Each data point is averaged over 10 times different realization corresponding to each N_h/N.

Fig.3 The plot of frequencies of cooperators as function of temptation to defect b corresponding to the different initial fractions of cooperators 20%, 50%, 80%, respectively. $N = 2001, \kappa = 2, m = 1000, N_h = 41$.

Fig.4 Frequency of cooperators as function of the fraction of hubs N_h/N, for different values of the temptation to defect b as shown in the legend. Note that when $b = 1$, the game is not a proper PD game.

Fig.5 The frequency of cooperators ρ_c as function of the parameter space $(b, N_h/N)$. The red region indicates the cooperation of high level corresponding to the intermediate N_h and b.

Fig.6 The frequencies of cooperators ρ_c vs. the temptation to defect b with different levels of heterogeneity.

Fig.7 Plot of the fraction of cooperators vs. N_h/N for different values of m with $b = 1.1$.

Fig.8 Plot of the results with update rules corresponding to average and accumulated payoff respectively. Fig. 8-A plots the case when $b = 1.1$ and Fig. 8-B with $b = 1.3$

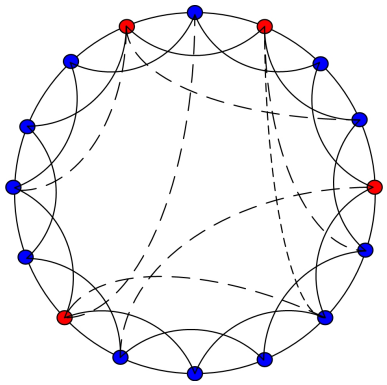

Fig. 1.

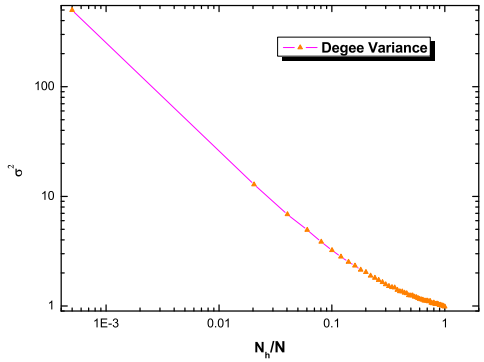

Fig. 2.

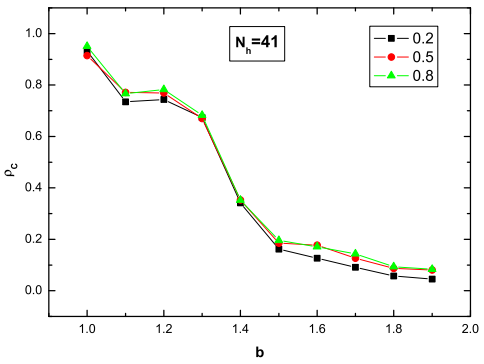

Fig. 3.

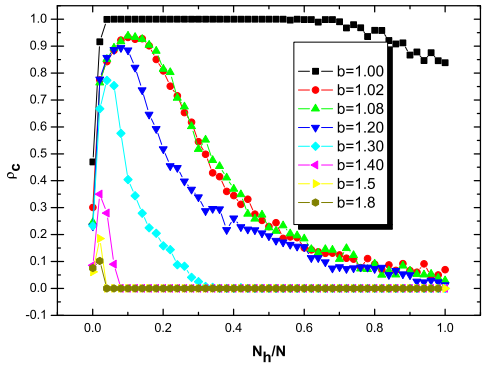

Fig. 4.

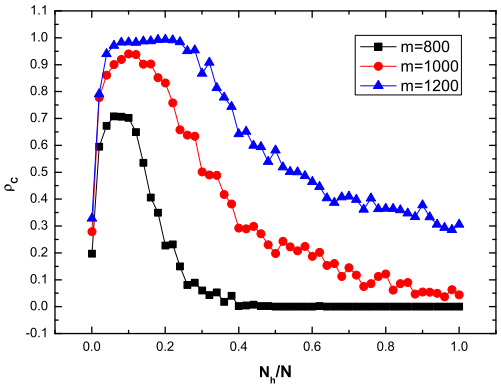

Fig. 7.

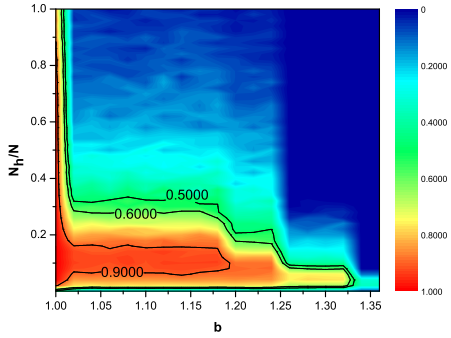

Fig. 5.

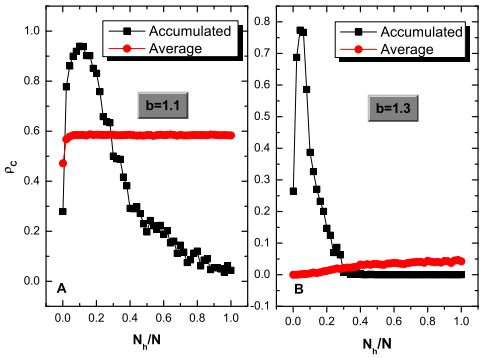

Fig. 8.

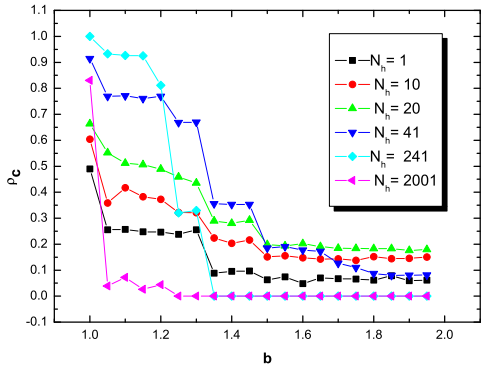

Fig. 6.